

Suborbits of a point stabilizer in the orthogonal group on the last subconstituent of orthogonal dual polar graphs

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Abstract

As one of the serial papers on suborbits of point stabilizers in classical groups on the last subconstituent of dual polar graphs, the corresponding problem for orthogonal dual polar graphs over a finite field of odd characteristic is discussed in this paper. We determine all the suborbits of a point-stabilizer in the orthogonal group on the last subconstituent, and calculate the length of each suborbit. Moreover, we discuss the quasi-strongly regular graphs and the association schemes based on the last subconstituent, respectively.

Keywords: orthogonal group, suborbit, dual polar graph, subconstituent, quasi-strongly regular graph, association scheme

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1. Introduction

Let \mathbb{F}_q be a finite field with q elements, where q is an odd prime power. Let \mathbb{F}_q^n be the row vector space of dimension n over \mathbb{F}_q . The set of all $m \times n$ matrices over \mathbb{F}_q is denoted by $M_{mn}(\mathbb{F}_q)$, and $M_{nn}(\mathbb{F}_q)$ is denoted by $M_n(\mathbb{F}_q)$ for simplicity. For any matrix $A = (a_{ij}) \in M_{mn}(\mathbb{F}_q)$, we denote the transpose of A by A^t .

Let $n = 2v + \delta$, where v is a non-negative integer and $\delta = 0, 1$ or 2 . Suppose

$$S_{2v+\delta,\Delta} = \begin{pmatrix} 0 & I^{(v)} \\ I^{(v)} & 0 \\ & \Delta \end{pmatrix}, \quad \Delta = \begin{cases} \phi \text{ (disappear)}, & \text{if } \delta = 0, \\ (1) \text{ or } (z), & \text{if } \delta = 1, \\ \begin{pmatrix} 1 & \\ & -z \end{pmatrix}, & \text{if } \delta = 2, \end{cases}$$

where z is a fixed non-square element of \mathbb{F}_q such that $1 - z$ is a non-square element. When $\delta = 1$ or 2 , Δ is definite in the sense that for any row vector $x \in \mathbb{F}_q^\delta$, $x\Delta x^t = 0$ implies $x = 0$. Note that the set

$$\left\{ T \in GL_{2v+\delta}(\mathbb{F}_q) \mid TS_{2v+\delta,\Delta}T^t = S_{2v+\delta,\Delta} \right\}$$

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forms a subgroup of $GL_{2\nu+\delta}(\mathbb{F}_q)$, called the *orthogonal group* of degree $n = 2\nu + \delta$ with respect to $S_{2\nu+\delta,\Delta}$ over \mathbb{F}_q , denoted by $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$. The group $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$ acts on $\mathbb{F}_q^{2\nu+\delta}$ by the matrix multiplication. $\mathbb{F}_q^{2\nu+\delta}$ together with this action is called the $(2\nu + \delta)$ -dimensional *orthogonal space* over \mathbb{F}_q with respect to $S_{2\nu+\delta,\Delta}$. A matrix representation of a subspace P is a matrix whose rows form a basis for P . When there is no danger of confusion, we use the same symbol to denote a subspace and its matrix representation. An m -dimensional subspace P of $\mathbb{F}_q^{2\nu+\delta}$ is called *totally isotropic* if $PS_{2\nu+\delta,\Delta}P^t = 0$. It is well-known that maximal totally isotropic subspaces of $\mathbb{F}_q^{2\nu+\delta}$ are of dimension ν .

Let G be a group acting transitively on a finite set X . For a fixed element $a \in X$, the stabilizer G_a is not transitive on X in general. The orbits of G_a on X are said to be *suborbits*, and the number of such suborbits is the *rank* of this action. H. Wei and Y. Wang [15, 16, 17] studied the suborbits of the transitive set of all totally isotropic subspaces under finite classical groups. We discussed these problems in singular classical spaces in [5, 12].

Dual polar graphs are famous distance-regular graphs and have been well studied ([1, 2, 10]). The *orthogonal dual polar graph* Γ (on the orthogonal space $\mathbb{F}_q^{2\nu+\delta}$) has as vertices the maximal totally isotropic subspaces; two vertices P and Q are adjacent if and only if $\dim(P \cap Q) = \nu - 1$. It is well-known that Γ is of diameter ν . For any vertex P of Γ , the i th subconstituent $\Gamma_i(P)$ with respect to P is the induced graph on the set of vertices at distance i from P in Γ . A. Munemasa [9] initiated the study of the subconstituents of dual polar graphs in the orthogonal spaces, and characterized the first and last subconstituents. Subsequently, Y. Wang, F. Li and Y. Huo [6, 7, 13, 14] characterized all the subconstituents of dual polar graphs under finite classical groups, and proved that for any vertex P of the dual polar graph Γ in the $(2\nu + \delta)$ -dimensional classical space (where $\delta = 0, 1$ or 2), the m th subconstituent $\Gamma_m(P)$ is isomorphic to $\begin{bmatrix} \nu \\ m \end{bmatrix}_q \cdot \mathcal{G}^{(m,\delta)}$, where $\mathcal{G}^{(m,\delta)}$ is the graph with the vertex set consisting of the matrices $\begin{pmatrix} X & Z \end{pmatrix}$ such that

$$\begin{cases} X + \overline{X}^t + ZI^{(\delta)}\overline{Z}^t = 0 & \text{the unitary case,} \\ X + X^t + Z\Delta Z^t = 0 & \text{the orthogonal case of odd characteristic,} \end{cases}$$

where $X \in M_m(\mathbb{F}_q)$, $Z \in M_{m\delta}(\mathbb{F}_q)$; and two vertices $\begin{pmatrix} X & Z \end{pmatrix}$ and $\begin{pmatrix} X_1 & Z_1 \end{pmatrix}$ are adjacent if and only if $\begin{pmatrix} X - X_1 & Z - Z_1 \end{pmatrix}$ is of rank 1. Note that the mapping

$$\begin{pmatrix} X & Z \end{pmatrix} \mapsto \begin{pmatrix} X & I^{(m)} & Z \end{pmatrix}$$

is an isomorphism from $\mathcal{G}^{(m,\delta)}$ to the last subconstituent of the corresponding dual polar graph in the classical space $\mathbb{F}_q^{2m+\delta}$. Therefore, the study of subconstituents of a dual polar graph may be reduced to that of the last subconstituent. In [8] we studied the suborbits of a point-stabilizer in the unitary group on the last subconstituent of Hermitean dual polar graphs. In this paper we discuss the corresponding problem for orthogonal dual polar graphs over a finite field of odd characteristic.

Let Γ be the orthogonal dual polar graph. It is well-known that a point-stabilizer of P of Γ in $O_{2\nu+\delta,\Delta}(\mathbb{F}_q)$ is transitive on the last subconstituent of Γ . In Section 2 we determine all the suborbits of this action, and calculate the rank and the lengths of these suborbits. As two applications of our results, in Sections 3 and 4, we discuss the quasi-strongly regular graphs and the association schemes based on the last subconstituent of Γ , respectively.

2. Suborbits

Let Γ be the dual polar graph in the orthogonal space $\mathbb{F}_q^{2\nu+\delta}$. Note that the last subconstituent Λ is a coclique when $\delta = 0$ (see [6]), and Λ is studied in [7] when $\delta = 1$. So the case $\delta = 2$ is the main objective of this paper.

Denote by $[X_1, X_2, \dots, X_t]$ the block diagonal matrix whose blocks along the main diagonal are matrices X_1, X_2, \dots, X_t , by $\mathcal{A}_{2r} = [\mathcal{A}_2, \dots, \mathcal{A}_2]$ the $2r \times 2r$ matrix of rank $2r$ in which $\mathcal{A}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and by $(A \ B \ \dots \ C)$ the matrix whose block entries are A, B, \dots, C . Suppose $I^{(m)}$ denotes the identity matrix of order m , and $0^{(p,q)}$ denotes the zero matrix of order p by q or $0^{(p)}$ when $p = q$.

We now study the suborbits of the stabilizer of each vertex P_0 in $O_{2\nu+2,\Delta}(\mathbb{F}_q)$ on Λ . Since $O_{2\nu+2,\Delta}(\mathbb{F}_q)$ acts transitively on the subspaces of the same type, we may choose $P_0 = (I^{(\nu)} \ 0^{(\nu)} \ 0^{(\nu,2)})$. By [6], Λ consists of subspaces of form $(A \ I^{(\nu)} \ Z)$, where $A \in M_\nu(\mathbb{F}_q)$ and $Z \in M_{\nu 2}(\mathbb{F}_q)$ satisfy $A + A^t + Z\Delta Z^t = 0$. Let G_0 be the stabilizer of P_0 in $O_{2\nu+2,\Delta}(\mathbb{F}_q)$. Then G_0 consists of matrices of the following form:

$$\begin{pmatrix} T_{11} & 0 & 0 \\ T_{21} & (T_{11}^t)^{-1} & T_{23} \\ -S\Delta T_{23}^t T_{11} & 0 & S \end{pmatrix},$$

where $T_{11} \in GL_\nu(\mathbb{F}_q)$, $T_{21} \in M_\nu(\mathbb{F}_q)$, $T_{23} \in M_{\nu 2}(\mathbb{F}_q)$, $S \in O_{2 \times 0 + 2, \Delta}(\mathbb{F}_q)$ and

$$(T_{11}^t)^{-1} T_{21}^t + T_{21} T_{11}^{-1} + T_{23} \Delta T_{23}^t = 0.$$

It is well-known that G_0 acts transitively on Λ . For any $P_1 \in \Lambda$, the suborbits of G_0 are just the orbits of the point-stabilizer of P_1 in G_0 on Λ . Let $P_1 = (0^{(\nu)} \ I^{(\nu)} \ 0^{(\nu,2)}) \in \Lambda$ and G_{01} be the stabilizer of P_0 and P_1 in $O_{2\nu+2,\Delta}(\mathbb{F}_q)$. Then G_{01} consists of matrices of the following form:

$$[T, (T^t)^{-1}, S], \quad (1)$$

where $T \in GL_\nu(\mathbb{F}_q)$ and $S \in O_{2 \times 0 + 2, \Delta}(\mathbb{F}_q)$. The action of $O_{2\nu+2,\Delta}(\mathbb{F}_q)$ on $\mathbb{F}_q^{2\nu+2}$ induces an action G_{01} on Λ :

$$\begin{aligned} \Lambda \times G_{01} &\longrightarrow \Lambda \\ ((A \ I^{(\nu)} \ Z), [T, (T^t)^{-1}, S]) &\longmapsto (T^t A T \ I^{(\nu)} \ T^t Z S). \end{aligned}$$

Denote by K_n the set of all $n \times n$ alternate matrices over \mathbb{F}_q . In order to determine the orbits of G_{01} on Λ , we need to introduce an action on K_ν . For $i = 1, 2$, let O_i denote the set of all matrices of the form

$$\begin{pmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{pmatrix}, \quad (2)$$

where $T_{11} \in GL_i(\mathbb{F}_q)$, $T_{21} \in M_{\nu-i,i}(\mathbb{F}_q)$ and $T_{22} \in GL_{\nu-i}(\mathbb{F}_q)$. Then O_i is a subgroup of $GL_\nu(\mathbb{F}_q)$, and there is an action of O_i on K_ν :

$$\begin{aligned} K_\nu \times O_i &\longrightarrow K_\nu \\ (X, T) &\longmapsto T^t X T. \end{aligned}$$

Note that $\{0^{(\nu)}\}$ is the trivial orbit of O_i on K_ν for $i = 1, 2$.

Theorem 2.1. (i) The nontrivial orbits of \mathcal{O}_1 on \mathbf{K}_v have the following representatives:

$$[0, \mathcal{A}_{2r}, 0^{(v-2r-1)}] \ (1 \leq r \leq \lfloor (v-1)/2 \rfloor), \quad [\mathcal{A}_{2r}, 0^{(v-2r)}] \ (1 \leq r \leq \lfloor v/2 \rfloor). \quad (3)$$

(ii) The nontrivial orbits of \mathcal{O}_2 on \mathbf{K}_v have the following representatives:

$$\begin{aligned} [0, \mathcal{A}_{2r}, 0^{(v-2r-1)}] \ (1 \leq r \leq \lfloor (v-1)/2 \rfloor), \quad [0^{(2)}, \mathcal{A}_{2r}, 0^{(v-2r-2)}] \ (1 \leq r \leq \lfloor (v-2)/2 \rfloor), \\ [\mathcal{A}_{2r}, 0^{(v-2r)}] \ (1 \leq r \leq \lfloor v/2 \rfloor), \quad [\mathcal{K}, \mathcal{A}_{2r-4}, 0^{(v-2r)}] \ (2 \leq r \leq \lfloor v/2 \rfloor), \end{aligned} \quad (4)$$

where $\mathcal{K} = \begin{pmatrix} 0 & I^{(2)} \\ -I^{(2)} & 0 \end{pmatrix}$.

PROOF. We only prove (ii), and (i) can be treated similarly. Let $X \in \mathbf{K}_v$ with $\text{rank } 2r > 0$. Write

$$X = \begin{pmatrix} x\mathcal{A}_2 & X_{12} \\ -X_{12}^t & X_{22} \end{pmatrix},$$

where $X_{12} \in M_{2, v-2}(\mathbb{F}_q)$ and $X_{22} \in \mathbf{K}_{v-2}$. Then $\text{rank } X_{22} = 2(r-i)$, $i = 0, 1$ or 2 . Hence there is a $T_{11} \in GL_{v-2}(\mathbb{F}_q)$ such that $T_{11}^t X_{22} T_{11} = [\mathcal{A}_{2(r-i)}, 0^{(v-2r+2i-2)}]$. Let

$$X_{12} T_{11} = \begin{pmatrix} 2r-2i & v-2r+2i-2 \\ Y_{12} & Y_{13} \end{pmatrix} \text{ and } T = \begin{pmatrix} I^{(2)} & & \\ & T_{11} & \\ & & I^{(v-2r+2i-2)} \end{pmatrix} \begin{pmatrix} I^{(2)} & & \\ -\mathcal{A}_{2(r-i)} Y_{12}^t & I^{(2r-2i)} & \\ & & I^{(v-2r+2i-2)} \end{pmatrix}.$$

Then $T \in \mathcal{O}_2$ and

$$T^t X T = \begin{pmatrix} x_1 \mathcal{A}_2 & 0 & Y_{13} \\ 0 & \mathcal{A}_{2(r-i)} & 0 \\ -Y_{13}^t & 0 & 0^{(v-2r+2i-2)} \end{pmatrix},$$

where $x_1 \mathcal{A}_2 = x\mathcal{A}_2 - Y_{12} \mathcal{A}_{2(r-i)} Y_{12}^t$.

Case 1: $\text{rank } X_{22} = 2r$. Then $x_1 = 0$, $Y_{13} = 0$ and $T^t X T = [0^{(2)}, \mathcal{A}_{2r}, 0^{(v-2r-2)}]$.

Case 2: $\text{rank } X_{22} = 2(r-1)$. Then $\text{rank } Y_{13} = 0$ or 1 . If $\text{rank } Y_{13} = 0$, then $x_1 \neq 0$, $T_1 = [x_1^{-1}, I^{(v-1)}] \in \mathcal{O}_2$ and $T_1^t T^t X T T_1 = [\mathcal{A}_{2r}, 0^{(v-2r)}]$. If $\text{rank } Y_{13} = 1$, then there exists a $T_{12} \in GL_2(\mathbb{F}_q)$ and $T_{13} \in GL_{v-2r}(\mathbb{F}_q)$ such that

$$T_{12} Y_{13} T_{13} = \begin{pmatrix} 0 & 0^{(v-2r-1)} \\ 1 & 0^{(v-2r-1)} \end{pmatrix}.$$

Let $x_2 \mathcal{A}_2 = x_1 T_{12} \mathcal{A}_2 T_{12}^t$ and

$$T_2 = \begin{pmatrix} T_{12}^t & & \\ & I^{(2r-2)} & \\ & & T_{13} \end{pmatrix} \begin{pmatrix} I^{(2)} & & \\ & 0 & I^{(2r-2)} \\ & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ x_2 & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} I^{(v-2r-1)} & & \\ & I^{(v-3)} & \end{pmatrix}.$$

Then $T T_2 \in \mathcal{O}_2$ and $(T T_2)^t X (T T_2) = [0, \mathcal{A}_{2r}, 0^{(v-2r-1)}]$.

Case 3: $\text{rank } X_{22} = 2(r-2)$. Then $\text{rank } Y_{13} = 2$; and so there exists a $T_{14} \in GL_{v-2r+2}(\mathbb{F}_q)$ such that $Y_{13} T_{14} = (I^{(2)} \ 0^{(2, v-2r)})$. Let

$$T_3 = \begin{pmatrix} I^{(2r-2)} & & \\ & T_{14} & \\ & & I^{(v-2r)} \end{pmatrix} \begin{pmatrix} I^{(2)} & & \\ & 0 & I^{(2r-4)} \\ & I^{(2)} & 0 \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & 1 \\ & -x_1 & 1 \end{pmatrix} \begin{pmatrix} I^{(v-3)} & & \\ & I^{(v-2r)} & \end{pmatrix}.$$

Then $TT_3 \in \mathcal{O}_2$ and $(TT_3)^t X(TT_3) = [\mathcal{K}, \mathcal{A}_{2(r-2)}, 0^{(v-2r)}]$.

Note that matrices of difference ranks can not be in the same orbit. Now we show that any two distinct matrices in (4) cannot fall into the same orbit of \mathcal{O}_2 . Otherwise, there exists a $T \in \mathcal{O}_2$, which is of the form (2), carrying $[0, \mathcal{A}_{2r}, 0^{(v-2r-1)}]$ to $[0^{(2)}, \mathcal{A}_{2r}, 0^{(v-2r-2)}]$, then $T_{22}^t [0, \mathcal{A}_{2(r-1)}, 0^{(v-2r-1)}] T_{22} = [\mathcal{A}_{2r}, 0^{(v-2r-2)}]$, which is impossible since T_{22} is nonsingular. Similarly, the left cases may be handled.

By above discussion, the desired result follows. \square

To determine the orbits of G_{01} on Λ , we need the following two lemmas.

Lemma 2.2. *Let $a, b \in \mathbb{F}_q$ with $(a, b) \neq (0, 0)$. Then there exists a $T \in \mathcal{O}_{2 \times 0+2, \Delta}(\mathbb{F}_q)$ such that the subspace $(a, b)T$ has the matrix representation of the form $(1, 0)$ or $(1, 1)$ corresponding to $a^2 - zb^2$ is a square element or not, respectively.*

PROOF. Note that (a, b) is of type $(1, 1, 0, 1)$ or $(1, 1, 0, z)$ in $\mathbb{F}_q^{2 \times 0+2}$ corresponding to $a^2 - zb^2$ being a square element or not, respectively. The result follows from [11, Theorem 6.4]. \square

Lemma 2.3. *Any element of $\mathcal{O}_{2 \times 0+2, \Delta}(\mathbb{F}_q)$ has one of the following forms*

$$\begin{pmatrix} x & y \\ yz & x \end{pmatrix}, \begin{pmatrix} x & y \\ -yz & -x \end{pmatrix}, \quad (5)$$

where $x^2 - y^2z = 1$.

PROOF. Let $T \in \mathcal{O}_{2 \times 0+2, \Delta}(\mathbb{F}_q)$ and write

$$T = \begin{pmatrix} x & y \\ u & v \end{pmatrix},$$

where $x^2 - y^2z = 1$, $xu - yvz = 0$ and $u^2 - v^2z = -z$. If $xyuv \neq 0$, then $u = x^{-1}yvz$ and $v^2 = x^2$, i.e., $v = \pm x$ and $u = \pm yz$. Then T has one of the form (5) with $x^2 - y^2z = 1$. If $xyuv = 0$, then T has one of the following forms

$$\pm I^{(2)}, \pm \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} 0 & y \\ y^{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & y \\ -y^{-1} & 0 \end{pmatrix}$$

with $-y^2z = 1$, which are of the form (5). \square

Pick a fixed subset Ω of \mathbb{F}_q^* such that $\mathbb{F}_q^* = \Omega \cup -\Omega$, where $-\Omega = \{-a | a \in \Omega\}$. Let E_i denote the v -dimensional column vector having 1 as its i -entry and other entries 0's. Similar to [6, Theorem 4.1], any element of Λ is of the form $(X - 2^{-1}Z\Delta Z^t \quad I^{(v)} \quad Z)$, where $X \in \mathbb{K}_v$ and $Z \in M_{v2}(\mathbb{F}_q)$. Note that $\{\varphi_0\} = \{P_1\}$ is the trivial orbit of G_{01} on Λ . We have

Theorem 2.4. *The nontrivial orbits of G_{01} on Λ have the following representatives:*

$$\varphi_1(r) = ([\mathcal{A}_{2r}, 0^{(v-2r)}] \ I^{(v)} \ 0^{(v,2)}) \quad (1 \leq r \leq \lfloor v/2 \rfloor), \quad (6)$$

$$\varphi_2(r, a) = ([-2^{-1}(1 - za^2), \mathcal{A}_{2r}, 0^{(v-2r-1)}] \ I^{(v)} \ (E_1 \ aE_1)) \quad (0 \leq r \leq \lfloor (v-1)/2 \rfloor), \quad (7)$$

$$\varphi_3(r, a) = ([\mathcal{A}_{2r}, 0^{(v-2r)}] + [-2^{-1}(1 - za^2), 0^{(v-1)}] \ I^{(v)} \ (E_1 \ aE_1)) \quad (1 \leq r \leq \lfloor v/2 \rfloor), \quad (8)$$

$$\varphi_4(r) = ([0, \mathcal{A}_{2r}, 0^{(v-1-2r)}] + [-2^{-1}\Delta, 0^{(v-2)}] \ I^{(v)} \ (E_1 \ E_2)) \quad (0 \leq r \leq \lfloor (v-1)/2 \rfloor), \quad (9)$$

$$\varphi_5(r) = ([Y, \mathcal{A}_{2r-2}, 0^{(v-1-2r)}] + [-2^{-1}\Delta, 0^{(v-2)}] \ I^{(v)} \ (E_1 \ E_2)) \quad (1 \leq r \leq \lfloor (v-1)/2 \rfloor), \quad (10)$$

$$\varphi_6(r) = ([-2^{-1}\Delta, \mathcal{A}_{2r}, 0^{(v-2r-2)}] \ I^{(v)} \ (E_1 \ E_2)) \quad (1 \leq r \leq \lfloor (v-2)/2 \rfloor), \quad (11)$$

$$\varphi_7(r, b) = ([b\mathcal{A}_2 - 2^{-1}\Delta, \mathcal{A}_{2r-2}, 0^{(v-2r)}] \ I^{(v)} \ (E_1 \ E_2)) \quad (1 \leq r \leq \lfloor v/2 \rfloor), \quad (12)$$

$$\varphi_8(r) = ([\mathcal{K}, \mathcal{A}_{2r-4}, 0^{(v-2r)}] + [-2^{-1}\Delta, 0^{(v-2)}] \ I^{(v)} \ (E_1 \ E_2)) \quad (2 \leq r \leq \lfloor v/2 \rfloor), \quad (13)$$

where $a \in \{0, 1\}$, $b \in \Omega$,

$$Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix},$$

and \mathcal{K} is given by Theorem 2.1. Moreover the rank of G_0 on Λ is

$$(q+7)/2 \cdot \lfloor v/2 \rfloor + 4 \cdot \lfloor (v-1)/2 \rfloor + \lfloor (v-2)/2 \rfloor + 3.$$

PROOF. Suppose $P \in \Lambda \setminus \{P_1\}$. Then $P = (X - 2^{-1}Z\Delta Z^t \ I^{(v)} \ Z)$, where $\text{rank}(X - 2^{-1}Z\Delta Z^t \ Z) > 0$.

If $Z = 0$, then $\text{rank } X = 2r > 0$, which implies that there exists a $T \in GL_v(\mathbb{F}_q)$ satisfying $T^t X T = [\mathcal{A}_{2r}, 0^{(v-2r)}]$. Observe $[T, (T^t)^{-1}, I^{(2)}] \in G_{01}$ carries P to (6).

If $Z \neq 0$, then $\text{rank } Z = 1$ or 2 . We distinguish the following two cases.

Case 1: $\text{rank } Z = 1$. Then there exists an $S \in GL_v(\mathbb{F}_q)$ such that $S^t Z = (xE_1 \ yE_1)$, where $(x, y) \neq (0, 0)$. By Theorem 2.1 there exists a $T \in \mathcal{O}_1$, which is of the form (2), such that $T^t(S^t X S)T$ is $0^{(v)}$ or of form (3). By Lemma 2.2, there exists an $S_{11} \in \mathcal{O}_{2 \times 0 + 2, \Delta}(\mathbb{F}_q)$ such that $T^t S^t Z S_{11} = b(E_1 \ aE_1)$, where $a \in \{0, 1\}$ and $b \in \mathbb{F}_q^*$. Observe

$$(ST)^t(Z\Delta Z^t)ST = (T^t S^t Z S_{11})\Delta(S_{11}^t Z^t S T) = b^2(E_1 \ aE_1)\Delta(E_1 \ aE_1)^t = [b^2(1 - za^2), 0^{(v-1)}].$$

Let $T_1 = [b^{-1}, I^{(v-1)}]$. Then $[S T T_1, ((S T T_1)^t)^{-1}, S_{11}] \in G_{01}$ carries P to

$$((S T T_1)^t(X - 2^{-1}Z\Delta Z^t)S T T_1 \ I^{(v)} \ (S T T_1)^t Z S_{11}).$$

Note that $\text{rank}((S T T_1)^t(X - 2^{-1}Z\Delta Z^t)S T T_1 \ (S T T_1)^t Z S_{11}) = \text{rank}(X - 2^{-1}Z\Delta Z^t \ Z)$.

If $(ST)^t X(S T) = 0^{(v)}$, then $[S T T_1, ((S T T_1)^t)^{-1}, S_{11}]$ carries P to (7) for $r = 0$.

If $(ST)^t X(S T) = [0, \mathcal{A}_{2r}, 0^{(v-1-2r)}]$, then $[S T T_1, ((S T T_1)^t)^{-1}, S_{11}]$ carries P to (7) for $r > 0$.

If $(ST)^t X(S T) = [\mathcal{A}_{2r}, 0^{(v-2r)}]$, then $[S T T_1, ((S T T_1)^t)^{-1}, S_{11}]$ carries P to

$$([b^{-1}\mathcal{A}_2, \mathcal{A}_{2r-2}, 0^{(v-2r)}] + [-2^{-1}(1 - za^2), 0^{(v-1)}] \ I^{(v)} \ (E_1 \ aE_1)).$$

Suppose $T_2 = [1, b, I^{(v-2)}]$. Then $[S T T_1 T_2, ((S T T_1 T_2)^t)^{-1}, S_{11}]$ carries P to (8).

Case 2: $\text{rank } Z = 2$. Then there exists an $S \in GL_v(\mathbb{F}_q)$ such that $S^t Z = (E_1 \ E_2)$. By Theorem 2.1, there exists a $T \in \mathcal{O}_2$, which is of the form (2) satisfying $T^t(S^t X S)T$ is $0^{(v)}$ or of form (4). Let $T_1 = [T_{11}^{-1}, I^{(v-2)}]$. Then $[S T T_1, ((S T T_1)^t)^{-1}, I^{(2)}] \in G_{01}$ carries P to

$$((S T T_1)^t(X - 2^{-1}Z\Delta Z^t)S T T_1 \ I^{(v)} \ (E_1 \ E_2)).$$

Observe

$$(STT_1)^t(Z\Delta Z^t)STT_1 = (T_1^t T^t S^t Z)\Delta(Z^t STT_1) = [\Delta, 0^{(v-2)}]$$

and

$$\text{rank}((STT_1)^t(X - 2^{-1}Z\Delta Z^t)STT_1 \begin{pmatrix} E_1 & E_2 \end{pmatrix}) = \text{rank}(X - 2^{-1}Z\Delta Z^t \begin{pmatrix} E_1 & E_2 \end{pmatrix}).$$

If $(ST)^t X(ST) = 0^{(v)}$, then $[STT_1, ((STT_1)^t)^{-1}, I^{(2)}]$ carries P to (9) for $r = 0$.

If $(ST)^t X(ST) = [0, \mathcal{A}_{2r}, 0^{(v-1-2r)}]$, then $[STT_1, ((STT_1)^t)^{-1}, I^{(2)}]$ carries P to

$$([Y_{u,v}, \mathcal{A}_{2r-2}, 0^{(v-1-2r)}] + [-2^{-1}\Delta, 0^{(v-2)}] \begin{pmatrix} I^{(v)} & \begin{pmatrix} E_1 & E_2 \end{pmatrix} \end{pmatrix}),$$

where

$$Y_{u,v} = \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & v \\ -u & -v & 0 \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix} = (T_{11}^{-1})^t \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Take $T_2 = [I^{(2)}, v^{-1}, I^{(v-3)}]$ or $[I^{(2)}, u^{-1}, I^{(v-3)}]$ according to $u = 0$ or not, respectively. Then $[STT_1 T_2, ((STT_1 T_2)^t)^{-1}, I^{(2)}]$ carries P to (9) for $r > 0$, or

$$([Y_{1,c}, \mathcal{A}_{2r-2}, 0^{(v-1-2r)}] + [-2^{-1}\Delta, 0^{(v-2)}] \begin{pmatrix} I^{(v)} & \begin{pmatrix} E_1 & E_2 \end{pmatrix} \end{pmatrix}),$$

where $c = u^{-1}v$. When $c^2 - z$ is a square element, we may choose an $s \in \mathbb{F}_q^*$ such that $c^2 - z = s^2$. Let $A = [A_{11}, s^{-1}, I^{(v-3)}]$, where

$$A_{11} = s^{-1} \begin{pmatrix} c & -z \\ -1 & c \end{pmatrix}.$$

By Lemma 2.3, $A_{11}^t \in O_{2 \times 0+2, \Delta}(\mathbb{F}_q)$, and $[A, (A^t)^{-1}, (A_{11}^t)^{-1}] \in G_{01}$ carries

$$([Y_{1,c}, \mathcal{A}_{2r-2}, 0^{(v-2r-1)}] + [-2^{-1}\Delta, 0^{(v-2)}] \begin{pmatrix} I^{(v)} & \begin{pmatrix} E_1 & E_2 \end{pmatrix} \end{pmatrix})$$

to (9) for $r > 0$. When $c^2 - z$ is a non-square element, we may choose an $s \in \mathbb{F}_q^*$ such that $s^2(c^2 - z) = 1 - z$. Let $B = [B_{11}, s, I^{(v-3)}]$, where

$$B_{11} = \frac{1}{s(c^2 - z)} \begin{pmatrix} c - z & z(c - 1) \\ c - 1 & c - z \end{pmatrix}.$$

By Lemma 2.3, $B_{11}^t \in O_{2 \times 0+2, \Delta}(\mathbb{F}_q)$, and $[B, (B^t)^{-1}, (B_{11}^t)^{-1}] \in G_{01}$ carries

$$([Y_{1,c}, \mathcal{A}_{2r-2}, 0^{(v-1-2r)}] + [-2^{-1}\Delta, 0^{(v-2)}] \begin{pmatrix} I^{(v)} & \begin{pmatrix} E_1 & E_2 \end{pmatrix} \end{pmatrix})$$

to (10).

If $(ST)^t X(ST) = [0^{(2)}, \mathcal{A}_{2r}, 0^{(v-2-2r)}]$, then $[STT_1, ((STT_1)^t)^{-1}, I^{(2)}]$ carries P to (11).

If $(ST)^t X(ST) = [\mathcal{A}_{2r}, 0^{(v-2r)}]$, then $[STT_1, ((STT_1)^t)^{-1}, I^{(2)}]$ carries P to $\varphi_7(r, b)$, where $(T_{11}^{-1})^t \mathcal{A}_2 T_{11}^{-1} = b \mathcal{A}_2$ for some $b \in \mathbb{F}_q^*$. Note that $[-1, I^{(v-1)}, -1, I^{(v-1)}, -1, 1] \in G_{01}$ carries $\varphi_7(r, b)$ to $\varphi_7(r, -b)$. So we may choose $b \in \Omega$.

If $(ST)^t X(ST) = [\mathcal{K}, \mathcal{A}_{2r-4}, 0^{(v-2r)}]$, then $[STT_1, ((STT_1)^t)^{-1}, I^{(2)}]$ carries P to

$$([U, \mathcal{A}_{2r-4}, 0^{(v-2r)}] + [-2^{-1}\Delta, 0^{(v-2)}] \begin{pmatrix} I^{(v)} & \begin{pmatrix} E_1 & E_2 \end{pmatrix} \end{pmatrix}),$$

where

$$U = \begin{pmatrix} 0 & (T_{11}^{-1})^t \\ -T_{11}^{-1} & 0 \end{pmatrix}.$$

Pick $T_3 = [I^{(2)}, T_{11}^t, I^{(\nu-4)}]$. Then $[S T T_1 T_3, ((S T T_1 T_3)^t)^{-1}, I^{(2)}]$ carries P to (13).

What is left to show that no two subspaces in (6) - (13) can fall into the same orbit. As an example, we show that any two distinct $\varphi_2(r, a)$ and $\varphi_2(r, a')$ can't fall into the same orbit, and the rest cases may be handled in a similar way. If there exists an element of G_{01} of form (1) carrying $\varphi_2(r, a)$ to $\varphi_2(r, a')$, then T is of the form

$$T = \begin{pmatrix} t & 0 \\ T_{21} & T_{22} \end{pmatrix},$$

where $t(1, a)S = (1, a')$. By [11, Theorem 6.4], the subspaces $(1, a)$ and $(1, a')$ is of the same type. Since $a, a' \in \{0, 1\}$, we have $a = a'$, a contradiction.

Therefore, the desired result follows. \square

For each vertex Q of Λ , the symbol \overline{Q} denotes the suborbit containing Q . By [11, Theorem 1.6, Theorem 3.16, Theorem 6.21], we have

$$\begin{aligned} |GL_\nu(\mathbb{F}_q)| &= q^{\nu(\nu-1)/2} \prod_{i=1}^{\nu} (q^i - 1), \\ |S p_{2\nu}(\mathbb{F}_q)| &= q^{\nu^2} \prod_{i=1}^{\nu} (q^{2i} - 1), \\ |O_{2\nu+2, \Delta}(\mathbb{F}_q)| &= q^{\nu(\nu+1)} \prod_{i=1}^{\nu} (q^i - 1) \prod_{i=0}^{\nu+1} (q^i + 1). \end{aligned}$$

Theorem 2.5. *The nontrivial orbits of G_{01} on Λ have lengths as following:*

$$\begin{aligned} |\overline{\varphi_1(r)}| &= \frac{|GL_\nu(\mathbb{F}_q)|}{|S p_{2r}(\mathbb{F}_q)| \cdot |GL_{\nu-2r}(\mathbb{F}_q)| \cdot q^{2r(\nu-2r)}}, \\ |\overline{\varphi_2(r, a)}| &= \frac{(q+1)|GL_\nu(\mathbb{F}_q)|}{|S p_{2r}(\mathbb{F}_q)| \cdot |GL_{\nu-2r-1}(\mathbb{F}_q)| \cdot 2q^{(2r+1)(\nu-2r-1)}}, \\ |\overline{\varphi_3(r, a)}| &= \frac{(q+1)|GL_\nu(\mathbb{F}_q)|}{|S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{\nu-2r}(\mathbb{F}_q)| \cdot 2q^{2r(\nu-2r)+2r-1}}, \\ |\overline{\varphi_4(r)}| &= \frac{(q+1) \cdot |GL_\nu(\mathbb{F}_q)|}{|S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{\nu-2r-1}(\mathbb{F}_q)| \cdot 2q^{(2r+1)(\nu-2r)-2}}, \\ |\overline{\varphi_5(r)}| &= \frac{(q+1)|GL_\nu(\mathbb{F}_q)|}{|S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{\nu-2r-1}(\mathbb{F}_q)| \cdot 2q^{(2r+1)\nu-4(r^2+1)}}, \\ |\overline{\varphi_6(r)}| &= \frac{|GL_\nu(\mathbb{F}_q)|}{|S p_{2r}(\mathbb{F}_q)| \cdot |GL_{\nu-2r-2}(\mathbb{F}_q)| \cdot q^{(2r+2)(\nu-2r-2)}}, \\ |\overline{\varphi_7(r, b)}| &= \frac{2|GL_\nu(\mathbb{F}_q)|}{|S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{\nu-2r}(\mathbb{F}_q)| \cdot q^{2r(\nu-2r)}}, \\ |\overline{\varphi_8(r)}| &= \frac{|GL_\nu(\mathbb{F}_q)|}{|S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{\nu-2r}(\mathbb{F}_q)| \cdot q^{2r(\nu-2r)+4r-5}}. \end{aligned}$$

PROOF. We only calculate $|\overline{\varphi_3(r, a)}|$ and $|\overline{\varphi_7(r, b)}|$. The length of other suborbits may be computed in a similar way.

Let $G_3(r, a)$ be the stabilizer of $\varphi_3(r, a)$ in G_{01} , and let $[T, (T^t)^{-1}, S]$ be any element of $G_3(r, a)$. Then

$$T^t([\mathcal{A}_{2r}, 0^{(\nu-2r)}] - [2^{-1}(1 - za^2), 0^{(\nu-1)}])T = [\mathcal{A}_{2r}, 0^{(\nu-2r)}] - [2^{-1}(1 - za^2), 0^{(\nu-1)}]$$

and $T'(E_1 \ aE_1)S = (E_1 \ aE_1)$, which imply that $\mu(1 \ a)S = (1 \ a)$ and

$$T = \begin{pmatrix} 1 & 1 & 2r-2 & v-2r \\ \mu & 0 & 0 & 0 \\ t & \mu & -\mu T_{31}^t \mathcal{A}_{2r-2} T_{33} & 0 \\ T_{31} & 0 & T_{33} & 0 \\ T_{41} & T_{42} & T_{43} & T_{44} \end{pmatrix}_{v-2r}^1, \quad \begin{matrix} 1 \\ 1 \\ 2r-2 \\ v-2r \end{matrix}$$

where $\mu^2 = 1$ and $T_{33}^t \mathcal{A}_{2r-2} T_{33} = A_{2r-2}$. By Lemma 2.3, $\mu(1 \ a)S = (1 \ a)$ implies that S is one of the following forms

$$\begin{cases} \mu I^{(2)}, \mu \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \text{if } a = 0, \\ \mu I^{(2)}, \frac{\mu}{1-z} \begin{pmatrix} 1+z & 2 \\ -2z & -(1+z) \end{pmatrix} & \text{if } a = 1. \end{cases}$$

Hence $|G_3(r, a)| = |S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{v-2r}(\mathbb{F}_q)| \cdot 4q^{2r(v-2r)+2r-1}$ and

$$\overline{|\varphi_3(r, a)|} = [G_{01} : G_3(r, a)] = \frac{(q+1)|GL_v(\mathbb{F}_q)|}{|S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{v-2r}(\mathbb{F}_q)| \cdot 2q^{2r(v-2r)+2r-1}}.$$

Let $G_7(r, b)$ be the stabilizer of $\varphi_7(r, b)$ in G_{01} . Then $G_7(r, b)$ consists of matrices $[T, (T')^{-1}, (T_{11}^t)^{-1}]$, where

$$T = \begin{pmatrix} 2 & 2r-2 & v-2r \\ T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ T_{31} & T_{32} & T_{33} \end{pmatrix}_{v-2r}^2,$$

$T_{11}^t \in O_{2 \times 0+2, \Delta}(\mathbb{F}_q)$, $T_{11}^t \mathcal{A}_2 T_{11} = \mathcal{A}_2$ and $T_{22}^t \mathcal{A}_{2r-2} T_{22} = \mathcal{A}_{2r-2}$. By Lemma 2.3, the matrix T_{11}^t satisfying $T_{11}^t \in O_{2 \times 0+2, \Delta}(\mathbb{F}_q)$ and $T_{11}^t \mathcal{A}_2 T_{11} = \mathcal{A}_2$ is of the form

$$T_{11}^t = \begin{pmatrix} x & y \\ yz & x \end{pmatrix},$$

where $x^2 - y^2z = 1$. By [11, Lemma 1.28], the number of solutions of the equation $x^2 - y^2z = 1$ is $q+1$. Hence $|G_7(r, b)| = |S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{v-2r}(\mathbb{F}_q)| \cdot (q+1)q^{2r(v-2r)}$ and

$$\overline{|\varphi_7(r, b)|} = [G_{01} : G_7(r, b)] = \frac{2|GL_v(\mathbb{F}_q)|}{|S p_{2r-2}(\mathbb{F}_q)| \cdot |GL_{v-2r}(\mathbb{F}_q)| \cdot q^{2r(v-2r)}}.$$

□

3. Quasi-strongly regular graphs

As a generalization of strongly regular graphs, quasi-strongly regular graphs were discussed by W. Golightly, W. Haynworth and D.G. Sarvate [4] and F. Goldberg [3]. Let c_1, c_2, \dots, c_p be distinct non-negative integers. A connected graph of degree k on n vertices is *quasi-strongly regular* with parameters $(n, k, \lambda; c_1, c_2, \dots, c_p)$ if any two adjacent vertices have λ common neighbors, and any two non-adjacent vertices have c_i common neighbors for some i ($1 \leq i \leq p$).

Since Γ is a regular near polygon, the induced subgraph on Λ is edge regular, denoted by the same symbol Λ . Therefore, Λ is quasi-strongly regular. In this section we compute all the parameters of Λ .

Let $\Lambda(P)$ be the set of neighbors of P in Λ . Clearly, $|\Lambda(P) \cap \Lambda(Q)| = 0$ whenever $\dim(P + Q) > \nu + 2$. Note that for the vertex P_1 in Λ as in Section 2, the subspace $Q \in \Lambda$ satisfying $\dim(Q + P_1) = \nu + 2$ lies in the set

$$\overline{\varphi_1(1)} \cup \overline{\varphi_4(0)} \bigcup_{a \in \{0,1\}} \overline{\varphi_3(1,a)} \bigcup_{b \in \Omega} \overline{\varphi_7(1,b)}.$$

To study $|\Lambda(P) \cap \Lambda(Q)|$ for any two vertices P and Q with $\dim(P + Q) = \nu + 2$, by Theorem 2.4, it suffices to consider $|\Lambda(P_1) \cap \Lambda(Q)|$, where $Q \in \{\varphi_1(1), \varphi_3(1, a), \varphi_4(0), \varphi_7(1, b)\}$, $a \in \{0, 1\}$ and $b \in \Omega$.

Lemma 3.1. *For any vertex $R = (A - 2^{-1}C\Delta C^t \ I^{(\nu)} \ C)$ of Λ , the neighborhood of R is*

$$\Lambda(R) = \{(A - 2^{-1}(C\Delta C^t + D\Delta D^t + 2D\Delta C^t) \ I^{(\nu)} \ C + D) \mid D \in M_{\nu 2}(\mathbb{F}_q), \text{ rank } D = 1\}.$$

PROOF. Note that $\Lambda(R)$ consists of matrices with the form $(A - 2^{-1}C\Delta C^t + X \ I^{(\nu)} \ C + D)$, where $X \in M_{\nu}(\mathbb{F}_q)$, $D \in M_{\nu 2}(\mathbb{F}_q)$, $\text{rank}(X \ D) = 1$ and $X + X^t + C\Delta D^t + D\Delta C^t + D\Delta D^t = 0$. It follows that $\text{rank } D = 1$. So we may write $D = D_0(x \ y)$ and $X = D_0 W^t$, where $0 \neq D_0 \in M_{\nu 1}(\mathbb{F}_q)$, $(x, y) \neq (0, 0)$ and $W \in M_{\nu 1}(\mathbb{F}_q)$. Let $C = (C_1 \ C_2)$ and $TD_0 = E_1$ for some $T \in GL_{\nu}(\mathbb{F}_q)$. Then

$$E_1(T(W + xC_1 - yzC_2))^t + T(W + xC_1 - yzC_2)E_1^t + (x^2 - y^2z)E_1E_1^t = 0.$$

It follows that $T(W + xC_1 - yzC_2) = -2^{-1}(x^2 - y^2z)E_1$. So $W = -2^{-1}(x^2 - y^2z)D_0 - xC_1 + yzC_2$ and $X = -2^{-1}(D\Delta D^t + 2D\Delta C^t)$. The desired result follows. \square

Note that when $\nu = 1$, any element of Λ is of the form $(-2^{-1}(a^2 - zb^2) \ 1 \ (a \ b))$, where $a, b \in \mathbb{F}_q$. Then Λ is a clique with q^2 vertices.

Lemma 3.2. *Let P and Q be any two vertices of Λ with $\dim(P + Q) = \nu + 2$. If $\nu \geq 2$, then $|\Lambda(P) \cap \Lambda(Q)|$ is equal to 0, q^2 , $q^2 - 1$ or $q^2 + q$.*

PROOF. For any $Q \in \{\varphi_1(1), \varphi_3(1, a), \varphi_4(0), \varphi_7(1, b)\}$, it suffices to show that $|\Lambda(P_1) \cap \Lambda(Q)| = 0, q^2, q^2 - 1$ or $q^2 + q$. We only compute $|\Lambda(P_1) \cap \Lambda(\varphi_3(1, a))|$, and the others can be treated similarly.

Let $R \in \Lambda(P_1) \cap \Lambda(\varphi_3(1, a))$. From $R \in \Lambda(P_1)$ and Lemma 3.1 we know that R is of the form $R = (-2^{-1}D\Delta D^t \ I^{(\nu)} \ D)$, where $D \in M_{\nu 2}(\mathbb{F}_q)$ and $\text{rank } D = 1$. Again from $R \in \Lambda(\varphi_3(1, a))$ and Lemma 3.1 we know that

$$R = ([\mathcal{A}_2, 0^{(\nu-2)}] - [2^{-1}(1 - za^2), 0^{(\nu-1)}] - 2^{-1}(D_1\Delta D_1^t + 2D_1\Delta(E_1 \ aE_1)^t) \ I^{(\nu)} \ (E_1 \ aE_1) + D_1),$$

where $D_1 \in M_{\nu 2}(\mathbb{F}_q)$ and $\text{rank } D_1 = 1$. Therefore, $D = (E_1 \ aE_1) + D_1$ and

$$-2^{-1}D\Delta D^t = [\mathcal{A}_2, 0^{(\nu-2)}] - [2^{-1}(1 - za^2), 0^{(\nu-1)}] - 2^{-1}(D_1\Delta D_1^t + 2D_1\Delta(E_1 \ aE_1)^t).$$

It follows that $-2^{-1}(E_1 \ aE_1)\Delta D_1^t = [\mathcal{A}_2, 0^{(\nu-2)}] - 2^{-1}D_1\Delta(E_1 \ aE_1)^t$; and so D is of the form

$$D = \begin{pmatrix} c+1 & zad_2-2 & 0 & \cdots & 0 \\ a+d_1 & d_2 & 0 & \cdots & 0 \end{pmatrix}^t,$$

where $cd_2 - d_1(zad_2 - 2) = 0$. Observe the number of solutions (c, d_1, d_2) satisfying $cd_2 - d_1(zad_2 - 2) = 0$ is $q + (q - 1)q = q^2$. Hence $|\Lambda(P_1) \cap \Lambda(\varphi_3(1, a))| = q^2$. \square

Theorem 3.3. *Let $v \geq 2$. Then Λ is a quasi-strongly regular graph with parameters*

$$(q^{v(v+3)/2}, (q^v - 1)(q + 1), q^v + q^2 - q - 1; 0, q^2, q^2 - 1, q^2 + q).$$

PROOF. Since Λ consists of the vertices as the form $(X - 2^{-1}Z\Delta Z^t \ I^{(v)} \ Z)$, where X is a $v \times v$ alternate matrix, and $Z \in M_{v2}(\mathbb{F}_q)$, we have $n = q^{v(v+3)/2}$. By Theorem 2.5,

$$k = |\overline{\varphi_2(0, 0)}| + |\overline{\varphi_2(0, 1)}| = 2|\overline{\varphi_2(0, 0)}| = (q^v - 1)(q + 1).$$

Note that $\varphi_2(0, a) \in \Lambda(P_1)$. In order to compute the parameter λ , it suffices to compute the size of the common neighbors of P_1 and $\varphi_2(0, a)$. Let $R \in \Lambda(P_1) \cap \Lambda(\varphi_2(0, a))$. From $R \in \Lambda(P_1)$ and Lemma 3.1 we know that R is of the form $R = (-2^{-1}D\Delta D^t \ I^{(v)} \ D)$, where $D \in M_{v2}(\mathbb{F}_q)$ and $\text{rank } D = 1$. Similar to the proof of Case 2 in Lemma 3.2, D is of rank 1 with the form

$$D = \begin{pmatrix} c + 1 & azd_2 & \cdots & azd_v \\ d_1 + 1 & d_2 & \cdots & d_v \end{pmatrix}^t.$$

Observe the number of matrices D satisfying $(d_2, \dots, d_v) = (0, \dots, 0)$ and $(d_2, \dots, d_v) \neq (0, \dots, 0)$ are $q^2 - 1$ and $(q^{v-1} - 1)q$, respectively. So $\lambda = |\Lambda(P_1) \cap \Lambda(\varphi_2(0, a))| = q^v + q^2 - q - 1$. The rest parameters of Λ are listed in Lemma 3.2. \square

4. Association schemes

In this section we discuss the association scheme based on Λ when $v = 2$.

A d -class association scheme \mathfrak{X} is a pair $(X, \{R_i\}_{0 \leq i \leq d})$, where X is a finite set, and each R_i is a nonempty subset of $X \times X$ satisfying the following axioms:

- (i) $R_0 = \{(x, x) \mid x \in X\}$;
- (ii) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$, $R_i \cap R_j = \emptyset$ ($i \neq j$);
- (iii) ${}^t R_i = R_{i'}$ for some $i' \in \{0, 1, \dots, d\}$, where ${}^t R_i = \{(y, x) \mid (x, y) \in R_i\}$;
- (iv) for all $i, j, k \in \{0, 1, \dots, d\}$, there exists an integer $p_{ij}^k = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|$ for every $(x, y) \in R_k$.

The integers p_{ij}^k are called the *intersection numbers* of \mathfrak{X} , and $k_i (= p_{ii'}^0)$ is called the *valency* of R_i . Furthermore, \mathfrak{X} is called *symmetric* if $i' = i$ for all i . As for more information concerning association schemes, the readers may consult [1, 2].

Let G be a transitive permutation group on a finite set X , and R_0, R_1, \dots, R_d be the orbits of the induced action of G on $X \times X$. It is well known that $(X, \{R_i\}_{0 \leq i \leq d})$ is an association scheme ([1, §2.2]).

Note that the action of G_0 on $\Lambda \times \Lambda$ determines an association scheme. We shall discuss the association scheme in the case $v = 2$.

In the rest we always assume that $v = 2$. By Theorem 2.4, the orbits of G_{01} on Λ have the following representatives:

$$\varphi_0, \varphi_1(1), \varphi_2(0, a), \varphi_3(1, a), \varphi_4(0), \varphi_7(1, b),$$

where $a \in \{0, 1\}$, $b \in \Omega$. For the action of G_0 on $\Lambda \times \Lambda$, let $R_0, R_1, R_{2_a}, R_{3_a}, R_4, R_{5_b}$ denote the orbits containing (φ_0, φ_0) , $(\varphi_0, \varphi_1(1))$, $(\varphi_0, \varphi_2(0, a))$, $(\varphi_0, \varphi_3(1, a))$, $(\varphi_0, \varphi_4(0))$, $(\varphi_0, \varphi_7(1, b))$, respectively. Then $R_0, R_1, R_{2_a}, R_{3_a}, R_4, R_{5_b}$ are all the orbits of the action of G_0 on $\Lambda \times \Lambda$.

Let $G_{\varphi_1(1)}$ be the stabilizer of $\varphi_1(1)$ in G_{01} . Then $G_{\varphi_1(1)}$ consists of matrices with the form $[T, (T')^{-1}, S]$, where $T' \in Sp_2(\mathbb{F}_q)$ and $S \in O_{2 \times 0+2, \Delta}(\mathbb{F}_q)$. So

$$|G_{\varphi_1(1)}| = |Sp_2(\mathbb{F}_q)| \cdot |O_{2 \times 0+2, \Delta}(\mathbb{F}_q)| = 2q(q-1)(q+1)^2.$$

In order to discuss the association scheme based on Λ , we need the following lemmas.

Lemma 4.1. *The orbits of $G_{\varphi_1(1)}$ on Λ have the following representatives:*

$$\begin{aligned}\phi_{1x} &= (x\mathcal{A}_2 \ I^{(2)} \ 0^{(2)}), \\ \phi_{2x,a} &= (x\mathcal{A}_2 + [-\frac{1}{2}(1-za^2), 0] \ I^{(2)} \ (E_1 \ aE_1)), \\ \phi_{3x,c} &= (x\mathcal{A}_2 + [-\frac{1}{2}c^2, \frac{1}{2}z] \ I^{(2)} \ [c, 1]),\end{aligned}$$

where $a \in \{0, 1\}$, $x \in \mathbb{F}_q$ and $c \in \Omega$.

PROOF. The proof is similar to that of Theorem 2.4, and is omitted. \square

Lemma 4.2. *Let $c \in \Omega$. Then the number of (T, S) satisfying $T' \in Sp_2(\mathbb{F}_q)$, $S \in O_{2 \times 0+2, \Delta}(\mathbb{F}_q)$ and $T'[c, 1]S = [c, 1]$ is $q+1$.*

PROOF. Since $T'[c, 1]S = [c, 1]$, by Lemma 2.3,

$$T = \begin{pmatrix} \mu & c^{-2}sz \\ s & \mu \end{pmatrix}, S = \begin{pmatrix} \mu & -c^{-1}s \\ -c^{-1}sz & \mu \end{pmatrix},$$

where $\mu, s, c \in \mathbb{F}_q$ and $\mu^2 - c^{-2}s^2z = 1$. By [11, Lemma 1.28], the number of (μ, c) satisfying $\mu^2 - c^{-2}s^2z = 1$ is $q+1$, as desired. \square

Lemma 4.3. *The representatives $\phi_{1x}, \phi_{2x,a}, \phi_{3x,c}$ listed in Lemma 4.1 satisfy*

$$\begin{aligned}(\varphi_0, \phi_{10}) &\in R_0, & (\phi_{10}, \varphi_1(1)) &\in R_1, \\ (\varphi_0, \phi_{11}) &\in R_1, & (\phi_{11}, \varphi_1(1)) &\in R_0, \\ (\varphi_0, \phi_{1d}) &\in R_1, & (\phi_{1d}, \varphi_1(1)) &\in R_1, \\ (\varphi_0, \phi_{20,a}) &\in R_{2_a}, & (\phi_{20,a}, \varphi_1(1)) &\in R_{3_a}, \\ (\varphi_0, \phi_{21,a}) &\in R_{3_a}, & (\phi_{21,a}, \varphi_1(1)) &\in R_{2_a}, \\ (\varphi_0, \phi_{2d,a}) &\in R_{3_a}, & (\phi_{2d,a}, \varphi_1(1)) &\in R_{3_a}, \\ (\varphi_0, \phi_{30,c}) &\in R_4, & (\phi_{30,c}, \varphi_1(1)) &\in R_{5_{\varepsilon c^{-1}}}, \\ (\varphi_0, \phi_{31,c}) &\in R_{5_{\varepsilon c^{-1}}}, & (\phi_{31,c}, \varphi_1(1)) &\in R_4, \\ (\varphi_0, \phi_{3d,c}) &\in R_{5_{\varepsilon_1 c^{-1}d}}, & (\phi_{3d,c}, \varphi_1(1)) &\in R_{5_{\varepsilon_2 c^{-1}(1-d)}},\end{aligned}$$

where $d \in \mathbb{F}_q \setminus \{0, 1\}$, $\varepsilon, \varepsilon_1, \varepsilon_2 \in \{1, -1\}$, $\varepsilon c^{-1}, \varepsilon_1 c^{-1}d, \varepsilon_2 c^{-1}(1-d) \in \Omega$.

PROOF. We only show $(\varphi_0, \phi_{30,c}) \in R_4$ and $(\phi_{30,c}, \varphi_1(1)) \in R_{5_{\varepsilon c^{-1}}}$. The left cases may be treated similarly, and will be omitted. Note that $[c^{-1}, 1, c, 1, I^{(2)}] \in G_0$ carries φ_0 and $\phi_{30,c}$ to φ_0 and $\varphi_4(0)$, respectively, so $(\varphi_0, \phi_{30,c}) \in R_4$. Let $\varepsilon = 1$ or -1 according to $c^{-1} \in \Omega$ or $-c^{-1} \in \Omega$, respectively. Then

$$\begin{pmatrix} -c^{-1} & 0 & & & & \\ 0 & -\varepsilon & & & & \\ \frac{1}{2}c & 0 & -c & 0 & -c & 0 \\ 0 & -\frac{1}{2}\varepsilon z & 0 & -\varepsilon & 0 & -\varepsilon \\ -1 & 0 & & & 1 & 0 \\ 0 & \varepsilon z & & & 0 & \varepsilon \end{pmatrix} \in G_0$$

carries $\phi_{30,c}$ and $\varphi_1(1)$ to φ_0 and $\varphi_7(1, \varepsilon c^{-1})$, respectively, which implies $(\phi_{30,c}, \varphi_1(1)) \in R_{5_{\varepsilon c^{-1}}}$. \square

Theorem 4.4. *The configuration $\mathfrak{X} = (\Lambda, \{R_0, R_1, R_{2_a}, R_{3_a}, R_4, R_{5_b}\}_{a \in \{0,1\}, b \in \Omega})$ is a symmetric association scheme with class $(q+11)/2$, whose non-zero intersection numbers p_{ij}^1 are given by*

$$\begin{aligned} p_{01}^1 &= p_{10}^1 = 1, \quad p_{11}^1 = q-2, \quad p_{2_a, 3_a}^1 = p_{3_a, 2_a}^1 = (q-1)(q+1)^2/2, \\ p_{3_a, 3_a}^1 &= (q-2)(q-1)(q+1)^2/2, \quad p_{4, 5_b}^1 = p_{5_b, 4}^1 = p_{5_b, 5_{\varepsilon b d^{-1}(1-d)}}^1 = 2q(q^2-1), \end{aligned}$$

where $d \in \mathbb{F}_q \setminus \{0, 1\}$, $\varepsilon \in \{1, -1\}$ and $\varepsilon b d^{-1}(1-d) \in \Omega$.

PROOF. By Theorem 2.4, \mathfrak{X} forms an association scheme of class $(q+11)/2$.

Now we prove \mathfrak{X} is symmetric. Since

$$\begin{pmatrix} 1 & 0 & & \\ 0 & -1 & & \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{matrix} \\ \\ \\ I^{(2)} \end{matrix} \in G_0$$

interchanges $\varphi_1(1)$ and φ_0 , ${}^t R_1 = R_1$. The left cases can be treated similarly, and will be omitted.

In order to compute non-zero intersection numbers p_{ij}^1 of \mathfrak{X} , we need consider the cases listed in Lemma 4.3. Here we only calculate $p_{2_a, 3_a}^1$ and $p_{4, 5_b}^1$ by the way of examples.

Let $G_{\phi_{20,a}}$ be the stabilizer of $\phi_{20,a}$ in $G_{\varphi_1(1)}$. By Lemma 4.3, $p_{2_a, 3_a}^1 = [G_{\varphi_1(1)} : G_{\phi_{20,a}}]$, the index of $G_{\phi_{20,a}}$ in $G_{\varphi_1(1)}$. Note that $G_{\phi_{20,a}}$ consists of matrices $[T, (T^t)^{-1}, S]$, where

$$T^t \in Sp_2(\mathbb{F}_q), \quad T^t[1, 0]T = [1, 0], \quad S \in O_{2 \times 0 + 2, \Delta}(\mathbb{F}_q) \quad \text{and} \quad T^t(E_1 \quad aE_1)S = (E_1 \quad aE_1).$$

It follows that

$$T = \begin{pmatrix} \mu & 0 \\ t & \mu \end{pmatrix}, \quad S = \mu I^{(2)} \quad \text{or} \quad S = \frac{\mu}{1-z\alpha^2} \begin{pmatrix} 1+z\alpha^2 & 2\alpha \\ -2z\alpha & -(1+z\alpha^2) \end{pmatrix},$$

where $\mu^2 = 1$. Therefore, $|G_{\phi_{20,a}}| = 4q$ and

$$p_{2_a, 3_a}^1 = [G_{\varphi_1(1)} : G_{\phi_{20,a}}] = |G_{\varphi_1(1)}|/|G_{\phi_{20,a}}| = (q-1)(q+1)^2/2.$$

Let $G_{\phi_{30,c}}$ be the stabilizer of $\phi_{30,c}$ in $G_{\varphi_1(1)}$, where $c \in \Omega$. Then $G_{\phi_{30,c}}$ consists of matrices $[T, (T^t)^{-1}, S]$, where

$$T^t \in Sp_2(\mathbb{F}_q), \quad T^t[c^2, -z]T = [c^2, -z], \quad S \in O_{2 \times 0 + 2, \Delta}(\mathbb{F}_q) \quad \text{and} \quad T^t[c, 1]S = [c, 1].$$

Note that $T^t[c, 1]S = [c, 1]$ implies $T^t[c^2, -z]T = [c^2, -z]$. By Lemma 4.2, $|G_{\phi_{30,c}}| = q+1$; and by Lemma 4.3,

$$p_{4, 5_{\varepsilon c^{-1}}}^1 = [G_{\varphi_1(1)} : G_{\phi_{30,c}}] = |G_{\varphi_1(1)}|/|G_{\phi_{30,c}}| = 2q(q^2-1),$$

which is independent of choices of $c \in \Omega$. Hence $p_{4, 5_b}^1 = 2q(q^2-1)$. \square

Remarks. All the valencies of \mathfrak{X} are given by Theorem 2.5. By a similar method in this section, all the intersection numbers p_{ij}^k of \mathfrak{X} can be calculated.

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References

- [1] E. Bannai and T. Ito, *Algebraic Combinatorics I: Association Schemes*, The Benjamin/Cummings Publishing Company, Inc., Menlo Park, CA, 1984.
- [2] A. E. Brouwer, A. M. Cohn and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin, Heidelberg, 1989.
- [3] F. Goldberg, On quasi-strongly regular graphs, *Linear Multilinear Algebra* 54 (2006) 437–451.
- [4] W. Golightly, W. Haynworth and D.G. Sarvate, A family of connected quasi-strongly regular graphs, *Congress Numerantium*, 124 (1997) 89–95.
- [5] J. Guo and K. Wang, Suborbits of m -dimensional totally isotropic subspaces under finite singular classical groups, *Linear Algebra Appl.* 430 (2009) 2063–2069.
- [6] F. Li and Y. Wang, Subconstituents of dual polar graph in finite classical spaces III, *Linear Algebra Appl.* 349 (2002) 105–123.
- [7] F. Li and Y. Wang, A class of amply regular graphs related to the subconstituents of a dual polar graph, *Discrete Math.* 306 (2006) 2909–2915.
- [8] F. Li, K. Wang, J. Guo and J. Ma, Suborbits of a point stabilizer in the unitary group on the last subconstituent of Hermitean dual polar graphs, *Linear Algebra Appl.* 433 (2010) 333–341.
- [9] A. Munemasa, *The Geometry of Orthogonal Groups over Finite Fields*, Lecture Note in Mathematics, Vol. 3, Sophia University, Kioicho, Tokyo, Japan, 1996.
- [10] Z. Wan, Z. Dai, X. Feng and B. Yang, *Studies in Finite Geometry and the Construction of Incomplete Block Designs*, Science Press, Beijing 1966 (in Chinese).
- [11] Z. Wan, *Geometry of Classical Groups over Finite Fields* (2nd edition), Science Press, Beijing/New York, 2002.
- [12] K. Wang, J. Guo and F. Li, Suborbits of subspaces of type (m, k) under finite singular general linear group, *Linear Algebra Appl.* 431 (2009) 1360–1366.
- [13] Y. Wang, F. Li and Y. Huo, Subconstituents of dual polar graph in finite classical spaces II, *Southeast Asian Bull. Math.* 24 (2000) 643–654.
- [14] Y. Wang, F. Li and Y. Huo, Subconstituents of dual polar graph in finite classical spaces I, *Acta Math. Appl. Sinica* 24 (2001) 443–440 (in Chinese).
- [15] Y. Wang and H. Wei, Suborbits of the finite unitary group $U_n(F_{q^2})$ on the transitive set of subspaces of type $(s+1, 1)$, *Acta Math. Sinica* 36 (1993) 163–179 (in Chinese).
- [16] H. Wei and Y. Wang, Suborbits of the transitive set of subspaces of type $(m, 0)$ under finite classical groups, *Algebra Colloq.* 3 (1996) 73–84.
- [17] H. Wei and Y. Wang, Suborbits of the set of m -dimensional totally isotropic subspaces under actions of pseudosymplectic groups over finite fields of characteristic 2, *Acta Math. Sinica* 38 (1995) 696–707 (in Chinese).